Singularities of viscosity solutions and the stabilization problem in the plane

Ludovic Rifford

Singularities of Viscosity Solutions and the Stabilization Problem in the Plane

LUDOVIC RIFFORD

ABSTRACT. We study the general problem of globally asymptotically controllable affine systems in the plane. As preliminaries we present some results of independent interest. We study the regularity of some sets related to semiconcave viscosity supersolutions of Hamilton-Jacobi-Bellman equations. Then we deduce a construction of stabilizing feedbacks in the plane.

1. INTRODUCTION

This paper is concerned with the stabilization problem for a control system of the form

$$\dot{x} = f(x,\alpha) = f_0(x) + \sum_{i=1}^{m} \alpha_i f_i(x), \quad \alpha = (\alpha_1, \dots, \alpha_m) \in \bar{B}_m, \tag{1.1}$$

assuming that the maps $f_0, f_1, \dots, f_m$ are locally Lipschitz from $\mathbb{R}^2$ into $\mathbb{R}^2$. It is well known that, even if every initial state can be steered to the origin by an open-loop control $\alpha(t)$ $(t \in [0,\infty))$, there may not exist a continuous feedback control $\alpha = \alpha(x)$ which locally stabilizes the system (1.1). To solve this problem, one can have recourse to discontinuous stabilizing feedbacks. In recent years, two main approaches have been developed. The first relies on the construction of a control-Lyapunov function [11], the second is based on "patching" together suitable families of open loop controls [5]. In our previous papers [18] and [17], basing on the first approach we showed the interest of considering semiconcave control-Lyapunov functions and we derived very useful constructions of stabilizing feedbacks. The purpose of this paper is to build upon this work in order to provide specific results in the case of the stabilization problem in the plane. Considering a semiconcave control-Lyapunov function V for the system (1.1) we study in detail the nature of a certain set $\Sigma_\delta(V)$ which is included in the set of nondifferentiability of V. We prove it to be the union of a Lipschitz submanifold of the plane of

Indiana University Mathematics Journal ©, Vol. 52, No. 5 (2003)

dimension one and of a discrete set of points (closed in $\mathbb{R}^2 \setminus \{0\}$). Then we use this information to formulate the construction of a smooth stabilizing feedback outside $\overline{\Sigma_\delta(V)}$. Of course, this closed-loop feedback is not continuous. However, we are able to indicate and to classify the types of singularities that appear. In this way, we settle an open problem suggested by Bressan in [8].

Our paper is organized as follows: in Section 2 we develop some results about the regularity of singular sets of semiconcave functions. In Section 3, we present a possible classification of stabilizing feedbacks for globally asymptotically controllable control systems in the plane and we detail what happens in the case of one-dimensional control systems.

Throughout this paper, $\mathbb{R}$ denotes the set of real numbers, $\|\cdot\|$ a Euclidean norm of $\mathbb{R}^2$, B the open ball $\{x : \|x\| < 1\}$ in $\mathbb{R}^2$, $\bar{B}$ the closure of B and $B(x,r) = x + rB$ (resp. $\bar{B}(x,r) = x + r\bar{B}$) the ball (resp. the closed ball) centered at x and with radius r. Finally, $\mathrm{int}(A)$ will denote the interior of a set $A \subset \mathbb{R}^n$ and $\mathrm{co}(A)$ its convex hull.

2. SINGULARITIES OF VISCOSITY SOLUTIONS OF HJB EQUATIONS

The purpose of this section is to study the structure of the set of singularities of a certain class of semiconcave functions in the plane. Namely, we will show that if a semiconcave function is a supersolution of some Hamilton-Jacobi-Bellman equation in $\mathbb{R}^2$, then a part of its singularities constitutes a C^1-submanifold of the Euclidean space. Our approach is to take into account the equation itself to determine the structure of the singular set.

We study the functions $u : \Omega \to \mathbb{R}$ which are viscosity supersolutions of the following Hamilton-Jacobi-Bellman equation:

$$F(x,u,Du) = 0, \quad x \in \Omega, \tag{2.1}$$

where the Hamiltonian $F : \Omega \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function (Ω is an open set in $\mathbb{R}^2$) which is convex in the third variable. For the sake of brevity, we do not recall the different notions of viscosity solutions, we refer to [6, 7, 13]. Moreover, since we will take a "subdifferential" point of view, we refer to the usual references ([10, 12] and the viscosity bibliography) for the definitions of generalized gradients $\partial u(x)$, proximal subdifferentials $\partial_P u(x)$ (resp. superdifferentials $\partial^P u(x)$), and limiting gradients $D^* u(x)$ (that we also denoted by $\partial_L u(x)$ in previous papers). In particular, the reader may consult our previous paper [17] for such definitions. We proceed now to present briefly the notion of semiconcavity and its fundamental properties that will be needed in the sequel. We shall say that the function $u : \Omega \to \mathbb{R}$ is (locally) *semiconcave* on the open set Ω if for any point $x_0 \in \Omega$ there exist $\rho, C > 0$ such that

$$tu(x) + (1-t)u(y) - u(tx + (1-t)y) \leq Ct(1-t)\|x-y\|^2, \tag{2.2}$$

for all $x, y \in x_0 + \rho B$.

As it is easy to check, the property (2.2) amounts locally to the concavity of $x \mapsto u(x) - C\|x\|^2$. Hence, a semiconcave function g can be seen locally as the sum of a concave function and of a smooth function. This crucial point implies immediately the following property, which implies also that the generalized gradient of a semiconcave function equals the proximal and the viscosity superdifferentials (see [6, 12]). For any $x_0 \in \Omega$, for any $\zeta \in \partial u(x_0)$, we have

$$u(y) - u(x_0) - C\|y - x_0\|^2 \le \langle \zeta, y - x_0 \rangle \quad \forall y \in x_0 + \rho B, \tag{2.3}$$

where ρ and C are the constants given above.

In addition to that, the concavity of the function $u(\cdot) - C\|\cdot\|^2$ implies also the local monotonicity of the operator $x \mapsto -\partial u(x) + 2Cx$; it can be stated as follows: For any $x, y \in x_0 + \rho B$ we have

$$\forall \zeta_x \in \partial u(x),\ \forall \zeta_y \in \partial u(y), \quad \langle -\zeta_y + \zeta_x, y - x \rangle \ge -2C\|x - y\|^2. \tag{2.4}$$

Now if we assume that the semiconcave function u is a supersolution of (2.1), then we get that its limiting gradients satisfy:

$$\forall x \in \Omega,\ \forall \zeta \in D^* u(x), \quad F(x, u, \zeta) \ge 0. \tag{2.5}$$

We recall that the generalized gradient can be constructed from the limiting gradients as follows:

$$\partial u(x) = \mathrm{co}(D^* u(x)). \tag{2.6}$$

Property (2.5) will force the set of singularities of u to be relatively benign. Many works [2–4, 9] have been devoted to the study of the set of nondifferentiability of semiconcave functions. Among them Alberti, Ambrosio and Cannarsa [3] provided some upper bounds on the Hausdorff-dimension of singular sets of semiconcave functions.

Let $u : \Omega \to \mathbb{R}$ be a semiconcave function on an open set of $\mathbb{R}^2$. We define for any $k \in \{0, 1, 2\}$

$$\Sigma^k(u) := \{x \in \Omega \mid \dim(\partial u(x)) = k\}.$$

Clearly, $\Sigma^0(u)$ represents the set of differentiability of u and, moreover,

$$\Omega = \Sigma^0(u) \cup \Sigma^1(u) \cup \Sigma^2(u). \tag{2.7}$$

We will sometimes denote the singular set (i.e., $\bigcup_{k \ge 1} \Sigma^k(u)$) by $\Sigma(u)$, for the sake of simplicity.

In the paper cited above, the authors prove the following result. (We refer to [3, 4] for the proof and to the book of Morgan [16] for a serious survey of the Hausdorff dimension.)

Proposition 2.1. *The set $\Sigma^2(u)$ is a discrete of points and the set $\Sigma^1(u)$ has Hausdorff dimension ≤ 1.*

We are going to elaborate this result when the function u is itself a viscosity supersolution of (2.1). In order to do so, we set for any $x \in \Omega$,

$$\Psi(x) := \min_{\zeta \in \partial u(x)} F(x, u(x), \zeta).$$

Since the multivalued map $x \mapsto \partial u(x)$ is upper semicontinuous and F continuous, the function Ψ is lower semicontinuous from Ω into $\mathbb{R}$. Our first objective is to study for any nonnegative continuous function $\delta : \Omega \to \mathbb{R}$ the structure of the set

$$S_\delta(u) := \{x \in \Omega \mid \Psi(x) < -\delta(x)\} \subset \Sigma^1(u) \cup \Sigma^2(u).$$

We claim the following result.

Theorem 2.2. *Let Ω be an open set of $\mathbb{R}^2$ and $u : \Omega \to \mathbb{R}$ be a semiconcave viscosity supersolution of the Hamilton-Jacobi-Bellman equation (2.1). If δ is a nonnegative continuous function on Ω, then whenever it is not empty the set*

$$\Sigma_\delta(u) := \{\Sigma^1(u) \cap S_\delta(u)\} \setminus \{\overline{\Sigma^2(u)}\}$$

is a C^1-submanifold of Ω of dimension 1.

The proof of this result is principally based on the characterization of C^1 submanifolds given by Tierno in [19] and [20]. Let us present his result.

Let X be a given subset of $\mathbb{R}^N$. A unit vector $v \in \mathbb{R}^N$ is said to be *tangent* to X at p if there exists a sequence of elements $x_k \in X$ converging to p, such that $x_k \neq p$ and $(x_k - p)/\|x_k - p\| \to v$. The set

$$T_X(p) := \{\lambda v \text{ is tangent to } X \text{ at } p \text{ and } \lambda \geq 0\}$$

is called the *Bouligand tangent cone* to X at p. Furthermore, a unit vector $v \in \mathbb{R}^N$ is said to be *weakly tangent* to X at x if there exist two sequences of elements $x_k, y_k \in Z$, both converging to p, such that $x_k \neq y_k$ for every $k \in N$ and $(x_k - y_k)/\|x_k - y_k\| \to v$. The set

$$S_X(x) := \{\lambda v \mid v \text{ is weakly tangent to } X \text{ at } x \text{ and } \lambda \in \mathbb{R}\}$$

is called the *paratingent cone* of X at x. Tierno proved the following result.

Theorem 2.3. *Let $X \subset \mathbb{R}^N$. The set X is a C^1-submanifold of dimension p if and only if it is locally compact and for any $x \in X$, $T_X(x) = S_X(x) \simeq \mathbb{R}^p$.*

We proceed now to prove Theorem 2.2.

Proof of Theorem 2.2. We prove that $\Sigma_\delta(u)$ is a C^1-submanifold of dimension 1 in Ω whenever it is not empty. Let x_0 be fixed in $\Sigma_\delta(u)$; we shall prove that the Bouligand tangent cone and the paratingent cone to $\Sigma_\delta(u)$ at x_0 coincide.

First, since $\partial u(x_0)$ is one-dimensional, we can write it as the convex hull of two elements α and β. Define $\varphi : [0,1] \to \mathbb{R}$ as follows:

$$\varphi(t) = F(x_0, u(x_0), t\alpha + (1-t)\beta).$$

Since $\Psi(x_0) < -\delta(x_0)$, there exists $t_0 \in (0,1)$ such that $\varphi(t_0) < -\delta(x_0)$. Fix now q in $\partial u(x_0)^\perp$; we prove that $q \in T_{\Sigma_\delta}(x_0)$.

Applying Lemma 4.5 of [1] with q and $p = t_0\alpha + (1-t_0)\beta$, we get the existence of a Locally Lipschitz arc $x : [0,\sigma] \to B_R(x_0)$ (where σ is a constant depending only upon u), with $x(0) = x_0$ and such that

$$\lim_{s\to 0} \frac{x(s) - x_0}{s} = q, \quad \text{and}$$

$$p(s) := t_0\alpha + (1-t_0)\beta + \frac{x(s) - x_0}{s} - q \in \partial u(x(s)), \qquad \forall s \in [0,\sigma].$$

Since $F(x_0, u(x_0), t_0\alpha + (1-t_0)\beta) < -\delta(x_0)$, the continuity properties of F and δ ensure that for s small enough

$$F(x(s), u(x(s)), p(s)) < -\delta(x(s)).$$

On the other hand, since u is a viscosity supersolution of (2.1), we have that

$$\forall \zeta \in D^* u(x), \quad F(x, u(x), \zeta) \geq 0.$$

Hence we deduce that $\partial u(x(s))$ has at least dimension 1. Furthermore, since $x_0 \notin \overline{\Sigma^2(u)}$, we have that $x(s) \in \Sigma^1(u)$ for s small enough. This proves that $q \in T_{\Sigma_\delta(u)}(x_0)$; consequently the Bouligand tangent cone of $\Sigma_\delta(u)$ contains $\partial u(x_0)^\perp$. Let us now show that its paratingent cone is included in $\partial u(x_0)^\perp$. Let $(x_k)_k$ and $(y_k)_k$ be two sequences of $\Sigma_\delta(u)$ converging to x_0. Since for all k, $\Psi(x_k) < -\delta(x_k)$ and $\Psi(y_k) < -\delta(y_k)$ and since for the ends $\zeta^i_{x_k}$, $i = 1, 2$ of the segment $\partial u(x_k)$ (respectively $\partial u(y_k)$) we have $F(x_k, u(x_k), \zeta^i_{x_k}) \geq 0$, $i = 1$, 2 (respectively $F(y_k, u(y_k), \zeta^i_{y_k}) \geq 0$, $i = 1, 2$), we get by continuity of F and by its convexity in the third variable that the segment $\partial u(x_k)$ (respectively $\partial u(y_k)$) contains a subsegment (which is not a singleton) I_{x_k} (respectively I_{y_k}) such that

$$\lim_{k\to+\infty} I_{x_k} = \lim_{k\to+\infty} I_{y_k} = [\lambda, \mu] \subset [\alpha, \beta]$$

for the Hausdorff topology on compact sets. Now considering the monotonicity property of the operator $x \mapsto -\partial u(x) + 2Cx$, we get that for any k and for any couple $(\zeta_{x_k}, \zeta_{y_k})$ in $I_{x_k} \times I_{y_k}$

$$\langle -\zeta_{y_k} + \zeta_{x_k}, y_k - x_k \rangle \geq -2C\|y_k - x_k\|^2.$$

Since the limit subsegment $[\lambda, \mu]$ is not a singleton, we conclude easily that the paratingent cone to $\Sigma_\delta(u)$ at x_0 is included in $\partial u(x_0)^\perp$. To summarize, we proved that

$$\partial u(x_0)^\perp \subset T_{\Sigma_\delta(u)}(x_0) \subset S_{\Sigma_\delta(u)}(x_0) \subset \partial u(x_0)^\perp.$$

In order to apply Tierno's theorem it remains to show that our set $\Sigma_\delta(u)$ is locally compact.

First of all, for any $\bar{x}$ in $\Sigma_\delta(u)$ there exists a closed neighborhood $\mathcal{V}$ of $\bar{x}$ such that

$$\Sigma_\delta(u) \cap \mathcal{V} \subset \Sigma^1(u) \cap S_\delta(u).$$

On the other hand, since $\partial u(\bar{x})$ has dimension one, we can write it as

$$\partial u(\bar{x}) = [\zeta_1, \zeta_2],$$

with ζ_1 and ζ_2 in $D^* u(\bar{x})$. The convexity of $F(\bar{x}, u(\bar{x}), \cdot)$ implies that the set

$$F^{-1}(\bar{x}, u(\bar{x}), \cdot)([0, \infty)) \cap \partial u(\bar{x})$$

is the union of two disjoint segments I_1 and I_2 satisfying

$$\zeta_1 \in I_1 \quad \text{and} \quad \zeta_2 \in I_2.$$

In addition, $\Psi(\bar{x}) < -\delta(\bar{x})$ implies that there exists $\bar{\zeta} \in \partial u(\bar{x})$ such that

$$F(\bar{x}, u(\bar{x}), \bar{\zeta}) < -\delta(\bar{x}),$$

and inequality (2.5) implies that $D^* u(\bar{x}) \subset I_1 \cup I_2$. Thus we can cut $D^* u(\bar{x})$ into two closed parts $J_1 := I_1 \cap D^* u(\bar{x})$ and $J_2 := I_2 \cap D^* u(\bar{x})$ such that

$$\operatorname{co}(J_1) \cap \operatorname{co}(J_2) = \emptyset.$$

Now from the continuity of F in the three variables and the continuity of the function u, for any $\varepsilon > 0$, there exists a positive constant μ such that for any $\zeta \in \mathbb{R}^2$ with $\|\zeta - \bar{\zeta}\| < \mu$, for any x in $\bar{x} + \mu B$, we have

$$F(x, u(x), \zeta) < \Psi(\bar{x}) + \varepsilon, \tag{2.8}$$

and

$$\delta(\bar{x}) - \frac{\varepsilon}{2} < \delta(x) < \delta(\bar{x}) + \frac{\varepsilon}{2}. \tag{2.9}$$

On the other hand, the upper semicontinuity of $D^*(\cdot)$ implies the existence of a constant $\rho \in (0, \mu)$ such that for any $x \in \Omega$ with $\|x - \bar{x}\| < \rho$ we have

$$D^*u(x) \subset D^*u(\bar{x}) + \mu B.$$

Therefore if we consider $x \in \Sigma_\delta(u) \cap \mathcal{V}$ such that $\|\bar{x} - x\| < \rho$, then

$$\Psi(x) \leq \Psi(\bar{x}) + \varepsilon. \tag{2.10}$$

As a matter of fact, we know that $D^*u(x)$ is included in the union of the sets $J_1 + \mu B$ and $J_2 + \mu B$ and that $\Psi(x) < -\delta(x)$; this means that $D^*u(x)$ intersects both sets and consequently that $d(\bar{\zeta}, \partial u(x)) < \mu$, which implies directly $\Psi(x) < \Psi(\bar{x}) + \varepsilon$. Hence applying the inequalities (2.10), (2.8) and (2.9) with $\varepsilon = (-\Psi(\bar{x}) - \delta(\bar{x}))/2 > 0$, we get that

$$\Sigma_\delta(u) \cap \mathcal{V} \cap B(\bar{x}, \rho) = \left\{ x \in \mathcal{V} \cap B(\bar{x}, \rho) \mid \Psi(x) \leq -\delta(x) - \frac{\varepsilon}{2} \right\}. \tag{2.11}$$

Since the function Ψ is lower semicontinuous, we conclude from (2.11) that the set on the left in (2.11) is closed and hence that $\Sigma_\delta(u)$ is locally compact. The proof is complete. ◻

Remark 2.4. We notice that if we set $\tilde{\Sigma}_\delta(u) = \Sigma^1(u) \cap S_\delta(u)$, then we have, by the same arguments of the previous proof, that for any $x \in \tilde{\Sigma}_\delta$

$$T_{\tilde{\Sigma}_\delta(u)}(x) = S_{\tilde{\Sigma}_\delta(u)}(x) = \partial u(x)^\perp.$$

However, the set $\tilde{\Sigma}_\delta(u)$ is not locally compact, so we cannot deduce that it is a C^1-submanifold of the plane.

Let us notice that the multivalued map $x \mapsto \partial u(x)^\perp$ is continuous on the set $\Sigma_\delta(u)$. Furthermore, if we assume that the function u is a viscosity solution of (2.1), then the multivalued map $x \mapsto D^*u(x)$ (and *a fortiori* $x \mapsto \partial u(x)$) is continuous on the set $\Sigma_\delta(u)$. Of course the result of Theorem 2.2 does not hold in dimension $n \geq 2$. Actually the presence of $\Sigma^k(u)$ for $k \geq 3$ complicates the situation. Some precise analysis of the different parts of the sets $\Sigma^k(u)$ ($k \in [1, n]$) can lead to interesting results. Here we just present a one such result; one can adapt the preceding proof to obtain it.

Theorem 2.5. *Let Ω be an open set of $\mathbb{R}^N$, let $\delta : \mathbb{R}^N \to \mathbb{R}$ be a nonnegative continuous function and $u : \Omega \to \mathbb{R}$ be a semiconcave viscosity supersolution of the Hamilton-Jacobi-Bellman equation (2.1). If the Hamiltonian F is assumed to be continuous in the three variables and convex in p, and if moreover $\Sigma(u) = \Sigma^k(u)$ for some $k \in [1, N]$, then whenever the set $\Sigma(u) \cap S_\delta(u)$ is nonempty, it is a C^1-submanifold of Ω of dimension $N - k$.*

We can deduce as a corollary a general result concerning semiconcave viscosity solutions.

Corollary 2.6. *Let Ω be an open set of $\mathbb{R}^N$ and $u : \Omega \to \mathbb{R}$ be a semiconcave viscosity solution of the Hamilton-Jacobi-Bellman equation*

$$F(x, u, Du) = 0, \quad x \in \Omega.$$

If the Hamiltonian F is assumed to be continuous in the three variables and strictly convex in p, and if, moreover, $\Sigma(u) = \Sigma^k(u)$ for some $k \in [1, N]$, then whenever the set $\Sigma(u)$ is nonempty, it is a C^1-submanifold of Ω of dimension $N - k$.

Our objective now is to detail the shape of the set $S_\delta(u)$. In order to complete the result of Theorem 2.2 we need to introduce some new notations. Let us first begin by two lemmas.

Lemma 2.7. *Under the assumptions of Theorem 2.2, if $\bar{x} \in \Sigma^2(u)$, then*

$$\bigcap_{\mu>0} \overline{\partial u((\bar{x} + \mu\bar{B}) \setminus \{\bar{x}\})} \subset \partial\partial u(\bar{x}), \tag{2.12}$$

where $\partial\partial u(\bar{x})$ denotes the topological boundary of the set $\partial u(\bar{x})$.

Proof. Let ζ be an element of the left part of (2.12). There exists a sequence $(x_n)_n$ converging to $\bar{x}$ and a sequence $(\zeta_n)_n$ converging to ζ such that for any n, $\zeta_n \in \partial u(x_n)$. Now considering the monotonicity property of the operator $-\partial u(\cdot) + 2Cx$ (locally for some constant C), we get that for any $\bar{\zeta} \in \partial u(\bar{x})$,

$$\left\langle -\zeta_n + \bar{\zeta}, \frac{x_n - \bar{x}}{\|x_n - \bar{x}\|} \right\rangle \geq -2C\|x_n - \bar{x}\|.$$

Furthermore, if ζ does not belong to the boundary of $\partial u(\bar{x})$, then there exists a positive constant ρ such that $\zeta + \rho\bar{B} \subset \partial u(\bar{x})$. Hence applying the inequality above with $\bar{\zeta} = \zeta - \rho v$, where v denotes some cluster point of the sequence $(x_n - \bar{x})/\|x_n - \bar{x}\|)_n$, gives $-\rho\|v\|^2 \geq 0$. We deduce that the inclusion is true. □

Let $\bar{x} \in \Sigma^2(u)$; the topological boundary $\partial\partial u(\bar{x})$ of the generalized gradient $\partial u(\bar{x})$ is a closed path. We can look at the gradients $\zeta \in \partial\partial u(\bar{x})$ such that

$$F(\bar{x}, u(\bar{x}), \zeta) < -\delta(\bar{x});$$

let us denote by $\partial_\delta\partial u(\bar{x})$ the set of these gradients. We have the following lemma.

Lemma 2.8. *Under the assumptions of Theorem 2.2, if $\bar{x} \in \Sigma^2(u)$, then the set $\partial_\delta\partial u(\bar{x})$ has a finite number of connected components, each of which is a segment.*

Proof. If the set is empty, the lemma is vacuous. So let us assume that $\partial_\delta \partial u(\bar{x})$ is nonempty. Since this set is convex and compact, the Krein-Milman Theorem asserts that it is the convex hull of its extreme points. Moreover, since the function u is a supersolution of (2.1), any extreme point $\bar{\zeta}$ of $\partial u(\bar{x})$ satisfies

$$F(\bar{x}, u(\bar{x}), \bar{\zeta}) \geq 0.$$

The convexity of the map $p \mapsto F(\bar{x}, u(\bar{x}), p)$ implies immediatly that each connected component of $\partial_\delta \partial u(\bar{x})$ is a segment with length bounded below by a positive constant μ. Now if we consider a disc D containing $\partial u(\bar{x})$ such that its center belongs to $\operatorname{int}(\partial u(\bar{x}))$, then by the projection of $\partial_\delta \partial u(\bar{x})$ on the boundary of this disc (i.e., a circle), we deduce the existence of a polygon with vertices on the circle and with edges of length greater than some $\mu' > 0$. We deduce that the number of components of $\partial_\delta \partial u(\bar{x})$ is finite. $\square$

We can now partition the set $\Sigma^2(u)$ as follows:

$$\Sigma^2(u) = \bigcup_{k \in \mathbb{N}^*} \Sigma^2_{\delta,k}(u),$$

where $\Sigma^2_{\delta,k}(u) := \{x \in \Sigma^2(u) \mid \partial_\delta \partial u(x) \text{ has } k \text{ connected components}\}$.

Proposition 2.9. *Let Ω be an open set of $\mathbb{R}^2$ and $u : \Omega \to \mathbb{R}$ be a semiconcave viscosity supersolution of the Hamilton-Jacobi-Bellman equation* (2.1). *If δ is a nonnegative continuous function on Ω, then whenever it is not empty, the set*

$$\mathcal{E}_\delta(u) := \{\Sigma^1(u) \cup \Sigma^2_{\delta,2}(u)\} \cap S_\delta(u)$$

is a Lipschitz submanifold of Ω of dimension 1.

Proof. We have to prove that for any $\bar{x}$ in $\mathcal{E}_\delta(u)$ there exists a neighbourhood $\mathcal{V}$ of $\bar{x}$ in Ω such that $\mathcal{E}_\delta(u) \cap \mathcal{V}$ can be viewed as the image of the line $(-1, 1) \times \{0\}$ (in $\mathbb{R}^2$) by a bilipschitz homeomorphism from B_2 into $\mathcal{V}$. Let us consider $\bar{x} \in \mathcal{E}_\delta(u)$; different cases appear.

(a) $\bar{x} \in \Sigma_\delta(u)$. Then we know by Theorem 2.2 that there exists a neighbourhood $\mathcal{V}$ of $\bar{x}$ such that $\mathcal{V} \cap S_\delta(u)$ is a C^1 submanifold of dimension 1. Since $\bar{x} \notin \overline{\Sigma^2(u)}$, it can be separated from $\Sigma^2(u)$. Hence reducing if necessary the neighbourhood $\mathcal{V}$, we have that $S_\delta(u)$ and $\mathcal{E}_\delta(u)$ coincide in this set. We have the desired conclusion for $\bar{x}$.

(b) $\bar{x} \in \Sigma^2_{\delta,2}(u)$. Then for any connected component K_i $(i = 1, 2)$ of $\partial_\delta \partial u(\bar{x})$, there exists a unique $q_i \in \mathbb{R}^2$ satisfying

$$\exists p_i \in K_i, \ \forall p \in \partial u(\bar{x}), \quad \langle q_i, p - p_i \rangle \geq 0.$$

Therefore, as in the proof of Theorem 2.2, we can refer to Lemma 4.5 of [1] to associate with any connected component K_i a unique Lipschitz arc $x_i(\cdot)$: $[0,\sigma] \to \bar{x} + rB$ (where σ and r are some positive constants sufficiently small), with $x(0) = \bar{x}$ and such that

$$\lim_{s\to 0} \frac{x(s)-\bar{x}}{s} = q_i, \quad \text{and}$$
$$p_i(s) := p_i + \frac{x(s)-\bar{x}}{s} - q_i \in \partial u(x_i(s)), \quad \forall s \in [0,\sigma].$$

We claim that there exists $\mathcal{V}$, a neighbourhood of $\bar{x}$ in Ω, such that

$$\mathcal{E}_\delta(u) \cap \mathcal{V} = X_1 \cup X_2,$$

where the set X_1 (resp. X_2) denotes the graph of the arc $x_1(\cdot)$ (resp. $x_2(\cdot)$) on a small interval $[0,\tau]$ with $\tau > 0$. Let us prove this claim; we argue by contradiction.

Let us assume that there exists a sequence $(y_k)_k$ (in Ω), converging to $\bar{x}$, such that

$$\forall k, \quad y_k \in \mathcal{E}_\delta(u) \setminus \{X_1 \cup X_2\}.$$

By Lemma 2.7, if we consider a sequence $(\zeta_k)_{k\in N}$ of gradients in $\partial u(y_k)$ such that $F(y_k, u(y_k), \zeta_k) < -\delta(y_k)$, then any cluster point of this sequence belongs to $K_1 \cup K_2$. Hence considering a subsequence of $(\zeta_k)_{k\in\mathbb{N}}$ we can assume that $\limsup_{k\to\infty} \partial u(y_k) \subset K_1$. Furthermore, we remark that any cluster "point" of $(\partial u(y_k))_{k\in\mathbb{N}}$ for the Haussdorf topology is an interval with ends α and β (in K_1) satisfying

$$F(\bar{x}, u(\bar{x}), \alpha) \le -\delta(\bar{x}), \quad F(\bar{x}, u(\bar{x}), \beta) \le -\delta(\bar{x}). \tag{2.13}$$

For any $k \in \mathbb{N}$, we can set $P_k := \mathrm{proj}_X(y_k)$ the projection of y_k on the set $X := X_1 \cup X_2$. Since y_k does not belong to X, we have that $v_k = (y_k - P_k)/\|y_k - P_k\|$ is well-defined. Let us denote by v a cluster point of $(v_k)_{k\in\mathbb{N}}$ (we assume henceforth that $v_k \to v$). First of all, if there exists a subsequence $(y_\ell)_\ell$ (with $\ell \to \infty$) such that for any ℓ, $P_\ell = \bar{x}$, then we deduce by the semiconcavity of u that there exists a subsegment $[\alpha, \beta]$ of K_1 (with $\alpha \neq \beta$) satisfying that

$$\forall \zeta \in \partial u(\bar{x}),\ \forall \zeta' \in [\alpha,\beta], \quad \langle \zeta - \zeta', v\rangle \ge 0.$$

On the other hand, by construction of P_k, we have

$$\langle v, q_1\rangle \le 0 \quad \text{and} \quad \langle v, q_2\rangle \le 0.$$

The three inequalities and (2.13) lead to a contradiction (we let the reader prove this fact).

Consequently, we can assume that the projection P_k is always on the arc $X_1 \setminus \{\bar{x}\}$. Using the property of the projection and the semiconcavity of u, we get the existence of a subsegment $[\alpha, \beta]$ (with $\alpha \neq \beta$) of K_1 such that $\forall \zeta, \zeta'$

$$\langle v, q_1 \rangle = 0 \quad \text{and} \quad \langle \zeta - \zeta', v \rangle \geq 0.$$

Again we get a contradiction.

(c) If $\bar{x} \in \Sigma^1(u) \cap \overline{\Sigma^2(u)}$, then the situation is the same as in the previous case. We can construct two Lipschitz arcs around $\bar{x}$ and conclude by Remark 2.4. □

We are now able to detail the shape of the set $S_\delta(u)$.

Theorem 2.10. *Let Ω be an open set of $\mathbb{R}^2$ and $u : \Omega \to \mathbb{R}$ be a semiconcave viscosity supersolution of the Hamilton-Jacobi-Bellman equation (2.1). If δ is a nonnegative continuous function on Ω, then*

$$\overline{S_\delta(u)} = \overline{\mathcal{E}_\delta(u)} \cup \Big[\bigcup_{k \geq 3} \Sigma^2_{\delta,k}(u) \Big].$$

Moreover, the closure of $\mathcal{E}_\delta(u)$ is a locally finite union of Lipschitz ∂-submanifolds (i.e., manifolds with boundary) of Ω of dimension 1 and the set $\bigcup_{k \geq 3} \Sigma^2_{\delta,k}(u)$ is a closed discrete set of points.

The proof is left to the reader.

Remark 2.11. We notice that, like in the case of the elements of $\Sigma^2_{\delta,2}(u)$ (see the proof of Proposition 2.9), when $\bar{x}$ belongs to some $\Sigma^2_{\delta,k}(u)$ for some $k \geq 3$, any segment K_i can be associated with a Lipschitz arc $x_i(\cdot)$ such that

$$\lim_{s \to 0} \frac{x_i(s) - \bar{x}}{s} = q_i,$$

with

$$\langle q_i, p - p_i \rangle \geq 0, \qquad \forall p \in \partial u(\bar{x}),$$

and where p_i is some point of K_i.

3. The stabilization problem in the plane

3.1. The general statement. Let us assume now that the control system is affine in the control, that is

$$f(x, \alpha) = f_0(x) + \sum_{i=1}^{m} \alpha_i f_i(x), \qquad \forall (x, \alpha) \in \mathbb{R}^2 \times \bar{B}_m, \tag{3.1}$$

where the $f_0, \ldots, f_m$ are locally Lipschitz functions from $\mathbb{R}^2$ into $\mathbb{R}^2$ and where $\bar{B}_m$ is the closed unit ball of $\mathbb{R}^m$. Our study is concerned particularly with those systems which are globally asymptotically controllable at the origin (we refer for instance to our previous papers for the definition). We proved in [18] that such systems always admit a semiconcave control-Lyapunov function. Then we exploited this fact to produce in [17] a nice discontinuous stabilizing feedback. Let us recall briefly the course to be followed to get such a feedback from the existence of a semiconcave control-Lyapunov function. In order to be complete, let us before recall the definition of a control-Lyapunov function for the system (3.1).

Definition 3.1. A control-Lyapunov function (clf) for the system (3.1) is a continuous function $V : \mathbb{R}^2 \to \mathbb{R}$ which is positive definite (i.e., $V(0) = 0$ and $V(x) > 0$ for $x \neq 0$), proper (i.e., $V(x) \to \infty$ when $\|x\| \to \infty$), and which is a viscosity supersolution of

$$F(x, V(x), DV(x)) = 0, \tag{3.2}$$

with $F(x,u,p) := -u + \max_{\alpha \in A}\{-\langle f(x,\alpha), p\rangle\}$.

Let us assume from now on that the system (3.1) is globally asymptotically controllable. By the main result proved in our paper [18] there exists a control-Lyapunov function $V : \mathbb{R}^2 \to \mathbb{R}$ which is semiconcave on $\mathbb{R}^2 \setminus \{0\}$. Hence we are able to define as in [17] the following function

$$\Psi_V(x) := \min_{\alpha \in A} \max_{\zeta \in \partial V(x)} \langle \zeta, f(x,\alpha)\rangle. \tag{3.3}$$

This function is upper semicontinuous on $\Omega := \mathbb{R}^2 \setminus \{0\}$; moreover, we let the reader show that

$$\forall x \in \Omega, \quad \Psi(x) + \Psi_V(x) = -V(x), \tag{3.4}$$

where the function Ψ is the one associated with the Hamiltonian F in the previous section. Hence still using our notations, we obtain that for any nonnegative continuous function $\delta : \Omega \to \mathbb{R}$, we have

$$S_\delta(V) = \{x \in \Omega \mid \Psi_V(x) > \delta(x) - V(x)\}.$$

We will construct a feedback on the open set $\mathcal{D} := \mathbb{R}^2 \setminus [\overline{S_\delta(V)} \cup \{0\}]$. We omit the proof of the following proposition. A similar result is given in [17]; note that this proof is only based on Michael's theorem (see [15]).

Proposition 3.2. *Let E be an open dense set of $\mathbb{R}^2$ and $G : E \to 2^{\overline{B_m}}$ be a multivalued map. If for any x in E, $G(x)$ is nonempty and defined as follows*

$$G(x) = \{\alpha \in \overline{B_m} \mid \max_{\zeta \in \partial V(x)} \langle f(x,\alpha), \zeta\rangle \leq -\lambda(x)\},$$

where $\lambda : E \to \mathbb{R}$ is a lower semicontinuous function satisfying

$$\lambda(x) \in (0, -\Psi_V(x)), \quad \forall x \in E,$$

then there exists a continuous selection $\alpha : E \to \overline{B_m}$ such that

$$\forall x \in E, \quad \max_{\zeta \in \partial V(x)} \langle f(x, \alpha(x)), \zeta \rangle \leq -\lambda(x). \tag{3.5}$$

From this result, following the proof of Theorem 4 in [17], we can apply Proposition 3.2 with $\lambda := V/2$ (that is, $\delta := V/2$) and $E := \{x \in \mathbb{R}^2 \setminus \{0\} \mid \Psi_V(x) < -V(x)/2\}$. We get the following result.

Theorem 3.3. *Under the assumptions of Proposition 3.2, there exists a feedback $\alpha : \mathbb{R}^2 \to \overline{B_m}$, which is continuous on $\mathcal{D}$, for which the closed-loop system*

$$\dot{x}(t) = f(x(t), \alpha(x(t))) \tag{3.6}$$

is globally asymptotically stable in the sense of Carathéodory. Moreover, we have

$$\Psi_V(x(t)) \leq -\lambda(x(t)), \quad \forall t > 0, \tag{3.7}$$

along the Carathéodory trajectories of (3.6).

Of course, we encourage the reader to have a look at [17] for a good comprehension of this result.

3.2. One-dimensional control systems without drift. Let us assume that the control system is of the form

$$\dot{x} = \alpha g(x), \tag{3.8}$$

where the control α belongs to the interval $[a, b]$ and g is a C^2 vector field on the plane. Let there be given a one-dimensional system which is globally asymptotically controllable; our aim is to make precise the nature of a stabilizing feedback. We saw in [17] that such a feedback exists and that, in addition, it is repulsive for some closed set S. Here we will prove that the repulsive set S can be taken to be a C^1 submanifold of the state space of codimension 1. Since this result holds for any dimension, we will assume in the sequel that g is indeed C^2 from $\mathbb{R}^n$ into $\mathbb{R}^n$.

If the control $\alpha = 0$ does not belong to the interval $[a, b]$, then the globally asymptotically controllable system (3.8) is obviously globally stabilizable by a constant feedback (we let the reader prove this claim). Consequently, without loss of generality, we can assume that $[a, b] = [-1, 1]$. We have the following result.

Theorem 3.4. *If the system* (3.8) *is C^2, globally asymptotically controllable and not continuously stabilizable, then there exists a C^1-submanifold (without boundary) M of $\mathbb{R}^n \setminus \{0\}$ of codimension one and a stabilizing feedback $\alpha : \mathbb{R}^n \to [-1, 1]$ with*

$\alpha(0) = 0$ and $\alpha(x) = +1$ or -1 elsewhere such that the feedback α is continuous outside $M \cup \{0\}$. The feedback $\alpha(\cdot)$ is stabilizing in the sense of Carathéodory and moreover,

$$x(t) \notin M, \quad \forall t > 0.$$

In particular, the feedback is constant along the stabilizing trajectories for positive times.

Proof. We have to pass through an exact viscosity solution of some Hamilton-Bellman-Jacobi equation. Up to set $\tilde{V} := V^3$ we can assume that $V : \mathbb{R}^n \to \mathbb{R}$ is a semiconcave (outside the origin) viscosity supersolution of

$$-3V(x) + H(x, DV(x)) = 0, \quad x \neq 0. \tag{3.9}$$

Equivalently, for any $x \neq 0$, for any $\zeta \in \partial_L V(x)$,

$$\min_{\alpha \in [-1,1]} \langle \alpha g(x), \zeta \rangle \leq -3V(x).$$

Furthermore, we know by Theorem 2 of [17] that, for any $x_0 \in \mathbb{R}^n$, there exists a trajectory of (3.8) with $x(0) = x_0$ such that $V(x(t)) \leq V(x(0))e^{-3t}$. Therefore there exists a smooth (C^∞) positive definite function $W : \mathbb{R}^n \to \mathbb{R}$ satisfying

$$W(x(t)) \leq W(x(0))e^{-2t}, \tag{3.10}$$

for any $x_0 \in \mathbb{R}^n$ and any trajectory $x(\cdot)$ of (3.8) with $x(0) = x_0$. We define the following value function

$$v(x) := \inf_{\alpha \in \mathcal{A}} J(x, \alpha), \tag{3.11}$$

where the cost to minimize is

$$J(x, \alpha) := \int_0^{+\infty} e^t W(x_\alpha(t))\, dt,$$

where $x_\alpha(\cdot)$ denotes the trajectory of (3.8) which corresponds to the control α and with $x(0) = x_0$. Let us remark that by (3.10) the function v is well-defined; by classical arguments, it is a viscosity solution of

$$-v(x) + H(x, Dv(x)) = 0, \quad x \neq 0. \tag{3.12}$$

Moreover, since the dynamics g and the function W are sufficiently smooth (at least C^2), it can be shown that v is in fact semiconcave on $\mathbb{R}^N \setminus \{0\}$ (we refer to [6] or [14] for similar results).

We stress an outstanding property of the minimization problem associated with the value function v. Since our dynamics are one-dimensional, the reader will be able to prove that the control α remains constant almost everywhere (in t) along the minimizing trajectories of our minimization problem (of course some minimizing trajectories exist by (3.10)). Hence we deduce that we can see v as follows: $v(x) = \min\{v^+(x), v^-(x)\}$, where the functions v^+ and v^- are defined by

$$v^+(x) := \int_0^\infty e^t V(x_1(t))\,dt, \quad \text{and}$$
$$v^-(x) := \int_0^\infty e^t V(x_{-1}(t))\,dt;$$

the reader has probably understood that $x_1(\cdot)$ (resp. $x_{-1}(\cdot)$) refers to a trajectory corresponding to the control $\alpha \equiv 1$ (resp. $\alpha \equiv -1$).

These functions are C^1 where they are finite. We deduce that for any $x \in \mathbb{R}^n$,

$$\partial v(x) \subset \mathrm{co}(\nabla v^+(x), \nabla v^-(x)).$$

Thus $\Sigma(u) = \Sigma^1(u)$. Applying Theorem 2.5 of the second section, we get that $\Sigma^1(u)$ is a C^1 submanifold of dimension $n-1$. We conclude by Section 2.3 of [17]. $\square$

Remark 3.5. Theorem 3.4 holds in the case of a locally Lipschitz one dimensional control system. The proof is the same, although the function v is no longer semiconcave. Actually v is locally Lipschitz, so we get the existence of a submanifold of codimension 1 which is not C^1 but only locally Lipschitz.

Remark 3.6. In fact, by classical techniques, we can regularize the submanifold M into a closed smooth submanifold of $\mathbb{R}^2 \setminus \{0\}$.

3.3. Toward a possible classification. We return now to the general stabilization problem in the plane. We consider the affine control system (3.1) and the results that we developed in Section 3.1. In particular we have the control-Lyapunov function V verifying (3.2) and Theorem 3.3 gives us a stabilizing feedback α satisfying the conclusions of this result with $\lambda = V(x)/2$ and $E = \mathbb{R}^2 \setminus (S_\delta(V) \cup \{0\})$. We stress that the inequality (3.7) is satisfied along all the Carathéodory trajectories of the dynamical system (3.6). Let us classify the singularities in the manner conjectured by Bressan in [8]. Different types of points appear. (For the sake of simplicity we set $A := \overline{B_m}$.)

- *Let us consider* $\bar{x} \in \Sigma^2(V) \cap S_\delta(V)$ *isolated in* $S_\delta(V)$. Since $\bar{x}$ is isolated in $S_\delta(V)$, there exists a neighbourhood $\mathcal{V}$ (let us say a little ball centered at $\bar{x}$) of $\bar{x}$ such that

$$\forall x \in \mathcal{V} \setminus \{\bar{x}\}, \quad \Psi_V(x) \le \delta(x) - V(x). \tag{3.13}$$

If $\Psi_V(\bar{x}) < 0$, by setting in $\mathcal{V}$ a continuous function $\tilde{\delta}$ satisfying $\tilde{\delta}(\bar{x}) = \Psi_V(\bar{x}) + V(\bar{x}) \ge 0$ and $\tilde{\delta}(x) \ge \max\{0, \Psi_V(x) + V(x)\}$ for $x \neq \bar{x}$, we get that $\bar{x} \notin S_{\tilde{\delta}}(u)$

and consequently this singularity can be eliminated. So without loss of generality, we can assume that $\Psi_V(\bar{x}) \geq 0$ and that (3.13) is satisfied, i.e., for any x in $\mathcal{V} \setminus \{\bar{x}\}$, $\forall \zeta \in \partial V(x)$,

$$\langle \zeta, f(x, \alpha(x)) \rangle \leq \delta(x) - V(x) < 0,$$

which implies

$$\forall x \in \mathcal{V} \setminus \{\bar{x}\}, \quad \|f(x, \alpha(x))\| \geq \frac{V(x) - \delta(x)}{L_{\mathcal{V}}}, \tag{3.14}$$

where L_V denotes the Lipschitz constant of the function V in $\mathcal{V}$. Furthermore, let us remark that, by the technique of regularization used in Theorem 5 of [17], we can also assume that the vector field $f(x, \alpha(x))$ is smooth in the neighbourhood $\mathcal{V}$. We can now make precise the nature of the singularity $\bar{x}$; we claim the following lemma.

Lemma 3.7. *The convex compact set $f(\bar{x}, A)$ has dimension two, and* $0 \in \operatorname{int}(f(x, A))$.

Proof. We argue by contradiction. Let us assume that $\dim(f(\bar{x}, U)) = 1$.

This implies that it can be written as the convex hull of two vectors v_1 and v_2, i.e., $f(\bar{x}, A) = [v_1, v_2]$. Hence by continuity of the function f and by (3.14), the vector field $f(x, \alpha(x))$ takes values in two disjoint neighbourhoods of $[v_1, v_2] \cap \{v : \|v\| \geq (V(x) - \delta(x))/L_{\mathcal{V}}\}$. If the vector field takes values only in one of these two neighbourhoods on $\mathcal{V}$, then this implies that $\Psi_V(\bar{x}) \leq \delta(x) - V(x)$! So both sets are attained which is impossible by continuity of the flow of $f(x, \alpha(x))$ (recall that we assumed it to be smooth). Consequently we deduce that 0 can be separated from the convex set $[v_1, v_2]$; hence there exists $a \in \mathbb{R}^2 \setminus \{0\}$ and a positive constant μ such that

$$\forall v \in [v_1, v_2], \quad \langle v, a \rangle \geq \mu.$$

Thus by (3.7), the circle $\{x \in \mathcal{V} : \|x - \bar{x}\| = \rho\}$ (with ρ sufficiently small) is sent by the flow of $f(x, \alpha(x))$ into a closed path which is trivial in $\mathbb{R}^2 \setminus \{\bar{x}\}$. Since the circle that we consider is not contractible in $\mathbb{R}^2 \setminus \{\bar{x}\}$, we get a contradiction by continuity of the flow. Finally, we deduce that $f(\bar{x}, U))$ has dimension 2. Let us now prove the second part of the lemma.

Again we argue by contradiction. If $0 \notin \operatorname{int}(f(\bar{x}, A))$, then there exists a vector $a \in \mathbb{R}^2$ of norm 1 such that $0 \notin (a + f(\bar{x}, A))$. Hence by (3.14) there exists a constant μ such that

$$\forall x \in \mathcal{V}, \quad \langle f(x, \alpha(x)), a \rangle \geq \mu.$$

We conclude as before. $\square$

Finally, we conclude that $f(\bar{x}, A)$ is a two-dimensional compact convex set. In addition, we can regularize the field in a neighbourhood of $\bar{x}$ in order to obtain a field of the form $f(x, \alpha(x)) = \beta(x - \bar{x})$, where β is some positive constant corresponding to the size of $f(\bar{x}, A)$ (see Figure 3.1).

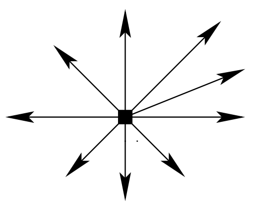

FIGURE 3.1. Repulsive point

• $\bar{x} \in \mathcal{E}_\delta(V)$. In this case the set $\mathcal{E}_\delta(V)$ can be seen locally as the image of the segment $(-1, 1)$ of the real line by a bilipschitz map. We claim the following lemma.

Lemma 3.8. *There exists a neighbourhood $\mathcal{V}$ of $\bar{x}$ and a positive constant ρ such that for any $x \in \mathcal{V}$, for any $s \in \mathrm{proj}_{\mathcal{E}_\delta(V)}(x)$,*

$$\langle x - s, f(x, \alpha(x)) \rangle \geq \rho \|x - s\|. \tag{3.15}$$

Proof. By (2.11), there exists a neighbourhood $\mathcal{V}$ (a ball centered at $\bar{x}$) of $\bar{x}$ such that

$$\Sigma_\delta(V) = \{x \in \mathcal{V} \mid \Psi(x) \leq -\delta(x) - \varepsilon\},$$

where ε is some positive constant. Hence by modifying the feedback α in $\mathcal{V}$ (using Proposition 3.2) we can assume that

$$\Psi_V(x) \geq -V(x) + \delta(x) + \varepsilon \quad \text{if } x \in \Sigma_\delta(V) \cap \mathcal{V}, \tag{3.16}$$

$$\max_{\zeta \in \partial V(x)} \langle \zeta, f(x, \alpha(x)) \rangle \leq -V(x) + \delta(x) \quad \text{if } x \in \mathcal{V} \setminus \Sigma_\delta(V). \tag{3.17}$$

On the other hand, since the set $\Sigma_\delta(V) \cap \mathcal{V}$ is a C^1 submanifold of the neighbourhood $\mathcal{V}$ and since $T^B_{\Sigma_\delta(V)} = \partial V(x)^\perp$ (we refer to Theorem 2.2), we have that for any $s \in \Sigma_\delta(V)$,

$$N^C_{\Sigma_\delta(V)}(s) = N^C_{\mathcal{E}_\delta(V)}(s) = \{t(\zeta_1(s) - \zeta_2(s)), t \in \mathbb{R}\}, \tag{3.18}$$

where $\partial V(s) = [\zeta_1(s), \zeta_2(s)]$. In fact, by Remark 2.4, we conclude also that for any $s \in \widetilde{\Sigma}_\delta(V)$,

$$\forall s \in \widetilde{\Sigma}_\delta(V), \quad N^C_{\mathcal{E}_\delta(V)}(s) = \{t(\zeta_1(s) - \zeta_2(s)), t \in \mathbb{R}\}. \tag{3.19}$$

Since $\mathcal{E}_\delta(V) \cap \mathcal{D}$ is a Lipschitz submanifold of dimension 1, it divides the ball $\mathcal{V}$ into two open connected components U_1 and U_2. Furthermore, by convexity of the Hamiltonian F and since V is a viscosity supersolution of (3.2), we get that if $x \in U_1$, then $\partial V(x)$ is included in some J_1 which is a neighbourhood of a connected component of $F(\bar{x}, V(\bar{x}), \cdot)^{-1}([0, +\infty))$ and if $x \in U_2$, then $\partial V(x)$ is included in the other component J_2. We conclude that if the ball $\mathcal{V}$ is chosen small enough, then for any x in U_1 such that $s = \mathrm{proj}_{\mathcal{E}_\delta(V)} \in \Sigma^1(V)$ we have

$$\begin{aligned} \langle \zeta_1(s), f(x, \alpha(x)) \rangle &\le \delta(x) - V(x) + \frac{\varepsilon}{4}, \quad \text{and} \\ \langle \zeta_2(s), f(x, \alpha(x)) \rangle &\ge \delta(x) - V(x) + \frac{3\varepsilon}{4}. \end{aligned}$$

Hence we compute for $x \in U_1$ such that $s := \mathrm{proj}_{\mathcal{E}_\delta(V)}(x) \in \Sigma^1(V)$

$$\begin{aligned} &\langle f(x, \alpha(x)), x - s \rangle \\ &\quad = \langle f(x, \alpha(x)), t(\zeta_2(s) - \zeta_1(s)) \rangle \qquad (t \ge 0, \text{ since } x \in U_1) \\ &\quad = -t\langle f(x, \alpha(x)), \zeta_1(s) \rangle + t\langle f(x, \alpha(x)), \zeta_2(s) \rangle \\ &\quad \ge -t\left(\delta(x) - V(x) + \frac{\varepsilon}{4}\right) + t\left(\delta(x) - V(x) + \frac{3\varepsilon}{4}\right) \\ &\quad \ge \frac{\varepsilon}{2}\left[\frac{\|x - s\|}{\|\zeta_2(s) - \zeta_1(s)\|}\right]. \\ &\quad \ge \frac{\varepsilon}{2\,\mathrm{diam}(\partial V(s))}\|x - s\|. \end{aligned}$$

Now let us consider x in U_1 such that $s := \mathrm{proj}_{\mathcal{E}_\delta(V)}(x)$ belongs to $\Sigma^2_{\delta,2}(V)$. Since x is in U_1, we have that $x - s \in N^P_{\overline{U_2}}(s)$. On the other hand, since $\overline{U_2}$ is epi-Lipschitz, we get by classical nonsmooth calculus (we refer to the book [12] for any nonsmooth notion) that

$$N^P_{\overline{U_2}}(s) \subset N^C_{\overline{U_2}}(s) = \mathrm{co}\{\lim_{s_k \to s} \zeta(s_k) \mid \zeta(s_k) \in N^C_{\overline{U_2}}(s_k) \text{ and } s_k \in \Sigma_\delta(V)\}.$$

This implies obviously that

$$N^C_{\overline{U_2}}(s) = \{t_1(\zeta_1' - \zeta_1) + t_2(\zeta_2' - \zeta_2) \mid t_1 \ge 0, t_2 \ge 0\},$$

where ζ_1 and ζ_1' (resp. ζ_2 and ζ_2') are in K_1 (resp. in K_2) and such that

$$\begin{aligned} \langle \zeta_1, f(x, \alpha(x)) \rangle &\le \delta(x) - V(x) + \frac{\varepsilon}{4}, \quad \text{and} \\ \langle \zeta_1', f(x, \alpha(x)) \rangle &\ge \delta(x) - V(x) + \frac{3\varepsilon}{4} \end{aligned}$$

(resp. replace ζ_1 and ζ_1' by ζ_2 and ζ_2'). We conclude as before and so we prove our lemma. $\square$

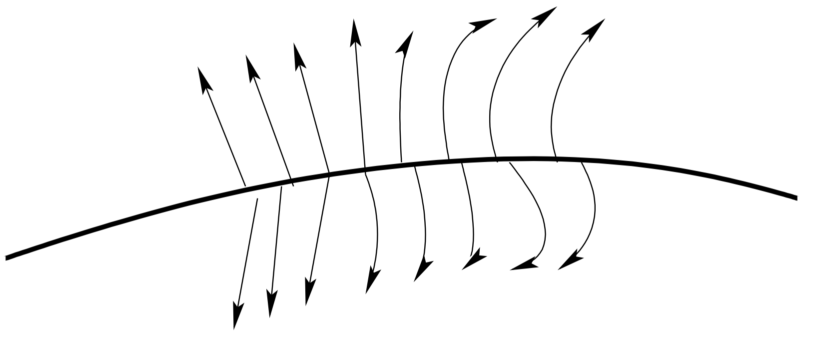

FIGURE 3.2. Around $\mathcal{E}_\delta(V)$

In fact, following the previous proof, we get that for any $\bar{x} \in \mathcal{E}_\delta(V)$, there exists a neighbourhood $\mathcal{V}$ and two controls α_1 and α_2 in A such that, if we denote by U_1 and U_2 the two connected components of $\mathcal{V} \setminus \mathcal{E}_\delta(V)$, we have

$$\forall x \in U_1,\ \forall \zeta \in \partial V(x), \quad \langle \zeta, f(x, \alpha_1) \rangle \leq \delta(x) - V(x) + \mu,$$

and

$$\forall x \in U_2,\ \forall \zeta \in \partial V(x), \quad \langle \zeta, f(x, \alpha_2) \rangle \leq \delta(x) - V(x) + \mu,$$

where μ is some positive constant. By (3.15), we deduce that there exists $\rho > 0$ such that for any y in $\mathcal{V} \cap \mathcal{E}_\delta(V) = S$, for any $\zeta \in N_S^C(y)$, there exists $i = 1, 2$ satisfying

$$\langle \zeta, f(x, \alpha_i) \rangle \geq \rho \|\zeta\|.$$

We are in the situation of Figure 3.2. We remark that, if we paste together all the α_i's which depend upon the $\bar{x}$'s (using a smooth partition of unity), we regularize our feedback $\alpha(\cdot)$ into a smooth feedback. This remark will enable us later to obtain a stabilizing feedback which will be smooth in our dense open set $\mathcal{D}$.

- $\bar{x} \in \bigcup_{k \geq 3} \Sigma^2_{\delta,k}(V)$. As in the first case, we prove the following lemma.

Lemma 3.9. *The convex compact set $f(\bar{x}, A)$ has dimension two.*

Proof. By definition, $\partial_\delta \partial V(\bar{x})$ has at least three components K_1, K_2, K_3. Moreover, on any K_i we have

$$\max_{\zeta \in K_i} \min_{\alpha \in A} \langle \zeta, f(x, \alpha) \rangle > \delta(x) - V(x) = -\frac{V(x)}{2}.$$

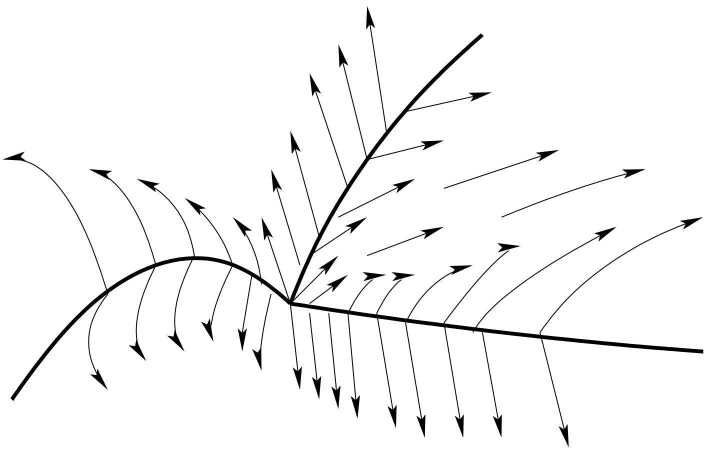

FIGURE 3.3. Repulsive multiple point

On the other hand, for the ends ζ_i^1 and ζ_i^2 of the segment K_i we have

$$\min_{\alpha \in A} \langle \zeta_i^1, f(x, \alpha) \rangle \leq -V(x).$$

Since $\partial V(\bar{x})$ has at least three extreme points (corresponding to the ends of the K_i's), we deduce that for any of them we can associate a velocity $f(\bar{x}, \alpha)$, that is at least three distinct velocities. Hence $f(\bar{x}, A)$ has dimension two. □

We explain what happens when $\bar{x}$ belongs to $\Sigma^2_{\delta,3}(V)$. In that case from Remark 2.11, each component K_i can be associated with a Lipschitz arc $x_i(\cdot)$. On the other hand, we know by Lemma 3.8 that if we are in some sufficiently small neighbourhood of some $x_i(t)$ ($t > 0$) then (3.15) holds. Hence we assert that there exists a continuous and even a smooth feedback α on some ball $\mathcal{V}$ centered at $\bar{x}$ (minus $\overline{S_\delta(V)}$) satisfying that for any $x \in \mathcal{V}$, for any $s \in \mathrm{proj}_{\overline{S_\delta(V)}}(x)$,

$$\langle x - s, f(x, \alpha(x)) \rangle \geq \rho \|x - s\|. \tag{3.20}$$

In view of the second case, we are in the situation of figure 3.3.

Remark 3.10. In the case of control system with scalar control:

$$\dot{x} = f_0(x) + \alpha_1 f_1(x),$$

the set of velocities is always a segment, hence it has dimension 1. In this case, it appears that repulsive multiple points cannot arise.

• $\bar{x} \in \partial\overline{\mathcal{I}_\delta(V)}$. From the proof of Theorem 2.10 we know that the Clarke tangent cone to $\overline{\mathcal{I}_\delta(V)}$ at $\bar{x}$ is a semiline which satisfies

$$T^C_{\overline{\mathcal{I}_\delta(V)}}(\bar{x}) \subset K^\perp,$$

where $K = \partial V(\bar{x})$ if $\bar{x} \in \Sigma^1(V)$ and $K = K_1$ if $\bar{x} \in \Sigma^2(V)$ (with the same notations as in the proof of Theorem 2.10). Therefore since $0 \notin \partial V(\bar{x})$ (because $\bar{x} \in \partial\overline{\mathcal{I}_\delta(V)}$) two cases will appear. Before explaining what happens, let us prove this result. for any $\zeta \in D^*V(\bar{x})$,

Lemma 3.11. *There exists $\bar{\alpha} \in A$ such that*

$$0 \neq f(\bar{x}, \bar{\alpha}) \in K^\perp_-, \tag{3.21}$$

where $K^\perp_- := \{p \in K^\perp \mid \langle \zeta, p\rangle \leq 0,\ \forall \zeta \in K\}$.

Proof. We treat the case where $\bar{x} \in \Sigma^1(V)$ and $K = \partial V(\bar{x})$; the other case $\bar{x} \in \Sigma^2(V)$ is left to the reader. Since $\bar{x}$ is on the boundary of $\overline{\mathcal{I}_\delta(V)}$, we have $\Psi_V(\bar{x}) = \delta(\bar{x}) - V(\bar{x})$. Hence we get the existence of $\alpha_0 \in A$ such that

$$\forall \zeta \in \partial V(\bar{x}, \langle \zeta, f(\bar{x}, \alpha_0)\rangle < 0.$$

On the other hand, there exists $\alpha \in A$ such that $\langle \zeta, f(\bar{x}, \alpha)\rangle \leq -V(\bar{x})$, for any $\zeta \in D^*V(\bar{x})$. Since $\partial V(\bar{x})$ is a segment, we can apply this property with both ends ζ_1, ζ_2 of the segment. That is, there exist two controls α_1 and α_2 in A such that

$$\langle \zeta_1, f(\bar{x}, \alpha_1)\rangle \leq -V(\bar{x}) \quad \text{and} \quad \langle \zeta_2, f(\bar{x}, \alpha_2)\rangle \leq -V(\bar{x}).$$

Actually we want to prove that $K^\perp_- \setminus \{0\}$ and $f(\bar{x}, A)$ have a nonempty intersection. But if $\langle \zeta_1 - \zeta_2, f(\bar{x}, \alpha_1)\rangle \geq 0$, this implies that

$$\langle \zeta_2, f(\bar{x}, \alpha_1)\rangle \leq \langle \zeta_1, f(\bar{x}, \alpha_1)\rangle \leq -V(x).$$

Thus we get that $\Psi_V(\bar{x}) \leq -V(\bar{x})$! This is impossible. Doing the same remark for α_2 we conclude that the convex set $\text{co}\{f(\bar{x}, \alpha_1), f(\bar{x}, \alpha_2), f(\bar{x}, \alpha_0)\}$ contains a control $\bar{\alpha}$ in A satisfying (3.21). ◻

Let us present the two cases.

(a) $T^C_{\overline{\mathcal{I}_\delta(V)}}(\bar{x}) = K^\perp_-$. By Lemma 3.11 there exists a neighbourhood $\mathcal{V}$ of $\bar{x}$ and a feedback $\alpha(\cdot) : \mathcal{V} \to A$ which is smooth on $\mathcal{V} \setminus \overline{S_\delta(V)}$ and such that $\alpha(\cdot)$ is continuous at $\bar{x}$ with $\alpha(\bar{x}) = \bar{\alpha}$. We are in the situation of an incoming cutedge (see Figure 3.4). In particular the feedback $\alpha(\cdot)$ can be taken such that there is only one trajectory starting from a point of $\mathcal{V} \setminus \overline{S_\delta(V)}$ which attains the singular set $\overline{S_\delta(V)}$.

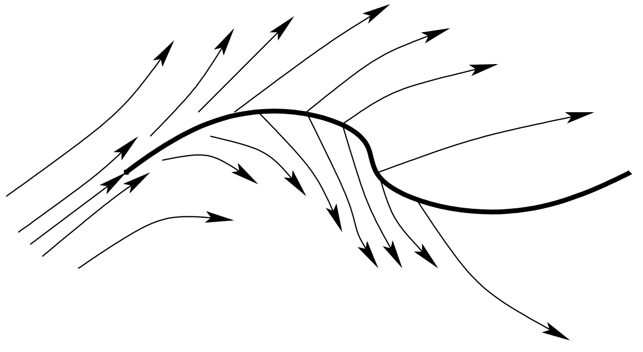

FIGURE 3.4. Incoming cutedge

(b) $T^C_{\overline{\Sigma_\delta(V)}}(\bar{x}) = -K^\perp_-$. By Lemma 3.11 there exists a neighbourhood $\mathcal{V}$ of $\bar{x}$ and a feedback $\alpha(\cdot) : \mathcal{V} \to A$ which is smooth on $\mathcal{V} \setminus \overline{S_\delta(V)}$ and such that $\alpha(\cdot)$ is continuous at $\bar{x}$ with $\alpha(\bar{x}) = \bar{\alpha}$. We are in the situation of an outgoing cutedge (see Figure 3.5). In this case the singular set is repulsive.

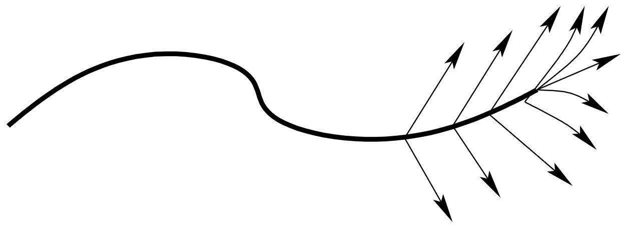

FIGURE 3.5. Outgoing cutedge

In conclusion, considering a feedback $\alpha(\cdot)$ given by Theorem 3.2 with $\lambda = V/2$ ($\equiv \delta = -V/2$), we proved that up to regularizing $\alpha(\cdot)$ we can assume that it is smooth on the open dense set $\mathcal{D}$ and that it can be extended continuously to the singularities of type "incoming cutedge". Hence we obtain a stabilizing feedback which displays only five types of singularities in $\mathbb{R}^2 \setminus \{0\}$.

REFERENCES

[1] P. ALBANO and P. CANNARSA, *Structural properties of singularities of semiconcave functions*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **28** (1999), 719–740.

[2] G. ALBERTI, *On the structure of singular sets of convex functions*, Calc. Var. Partial Differential Equations **2** (1994), 17–27.

[3] G. ALBERTI, L. AMBROSIO, and P. CANNARSA, *On the singularities of convex functions*, Manuscripta Math. **76** (1992), 421–435.

[4] G. AMBROSIO, P. CANNARSA, and H.M. SONER, *On the propagation of singularities of semi-convex functions*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **20** (1993), 597–616.

[5] F. ANCONA and A. BRESSAN, *Patchy vector fields and asymptotic stabilization*, ESAIM Control Optim. Calc. Var. (electronic) **4** (1999), 445–471.

[6] M. BARDI and I. CAPUZZO-DOLCETTA, *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations*, Birkhäuser Boston, Boston, MA, 1997.

[7] G. BARLES, *Solutions de viscosité des équations de Hamilton-Jacobi*, Springer-Verlag, Paris, 1994.

[8] A. BRESSAN, *Singularities of stabilizing feedbacks*, Proc. Control theory and its applications (Grado, 1998), vol. 56, 2001, pp. 87–104.

[9] P. CANNARSA and H.M. SONER, *On the singularities of the viscosity solutions to Hamilton-Jacobi-Bellman equations*, Indiana Univ. Math. J. **36** (1987), 501–524.

[10] F.H. CLARKE, *Optimization and Nonsmooth Analysis*, Classics in Applied Mathematics, vol. 5, Wiley-Interscience, New York, 1990.

[11] F.H. CLARKE, Yu.S. LEDYAEV, E.D. SONTAG, and A.I. SUBBOTIN, *Asymptotic controllability implies feedback stabilization*, I.E.E.E. Trans. Aut. Control **42** (1997), 1394–1407.

[12] F.H. CLARKE, Yu.S. YU.S. LEDYAEV, R.J. R.J. STERN, and P.R. P.R. WOLENSKI, *Nonsmooth Analysis and Control Theory*, Graduate Texts in Mathematics, vol. 178, Springer-Verlag, New York, 1998.

[13] M.G. CRANDALL and P.-L. LIONS, *Viscosity solutions of Hamilton-Jacobi equations*, Trans. Amer. Math. Soc. **277** (1983), 1–42.

[14] P.-L. LIONS, *Generalized Solutions of Hamilton-Jacobi Equations*, Pitman Advanced Publishing Program, Boston, Mass., 1982.

[15] E. MICHAEL, *Continuous selections I*, Ann. of Math. (2) **63** (1956), 361–382.

[16] F. MORGAN, *Geometric Measure Theory. A beginner's Guide*, Academic Press, Boston, MA, 1988.

[17] L. RIFFORD, *Semiconcave control-Lyapunov functions and stabilizing feedbacks*, SIAM J. Control Optim. **41** (2002), 659–681.

[18] ———, *Existence of Lipschitz and semiconcave control-Lyapunov functions*, SIAM J. Control Optim. **39** (2000), 1043–1064.

[19] G. TIERNO, *Una caratterizzazione geometrica della varietà differentiabili di* $\mathbf{R}^m$, Boll. Un. Mat. Ital. B (7) **11** (1997), 267–276.

[20] ———, *The paratingent space and a characterization of* C^1*-maps defined on arbitrary sets*, J. Nonlinear Convex Anal. **1** (2000), 129–154.

Institut Girard Desargues
Université Claude Bernard
Lyon I 69622 Villeurbanne, France
E-MAIL: `rifford@igd.univ-lyon1.fr`

KEY WORDS AND PHRASES: Asymptotic controllability, control-Lyapunov function, feedback stabilization, viscosity solutions.

2000 MATHEMATICS SUBJECT CLASSIFICATION: 93B20,93D05,93D20,49L25,70K15.

Received: *February 21st, 2002; revised: July 5th, 2002.*